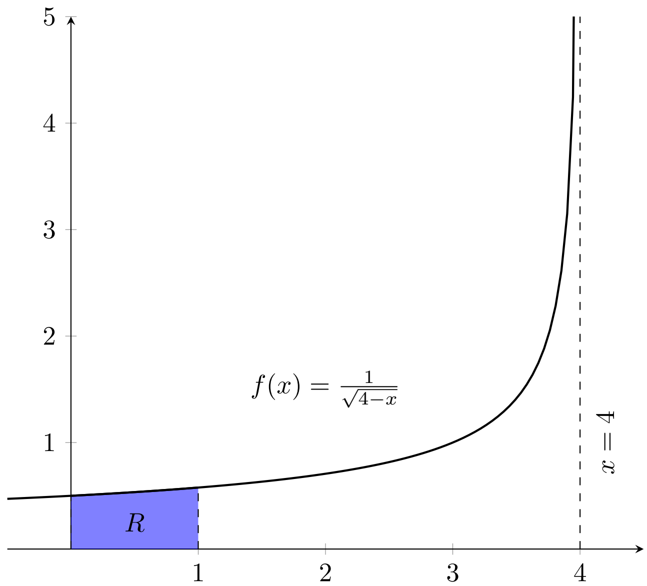
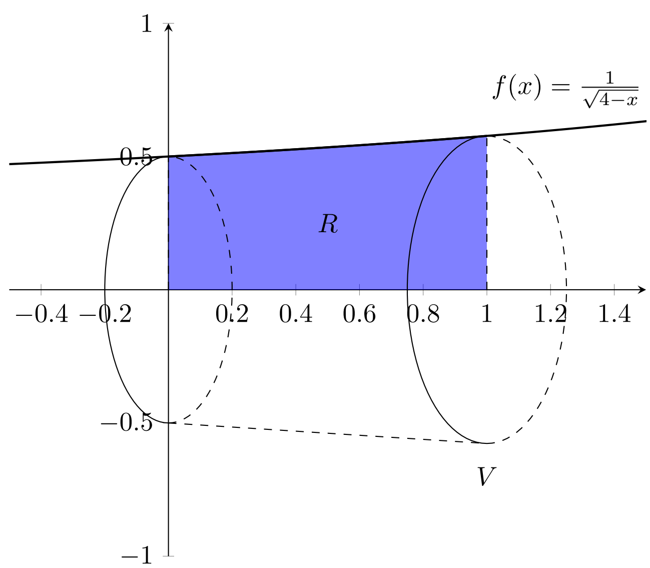
1 Review Problem 1 (from the Spring 2019 Final)

1. Let $f(x) = \frac{1}{\sqrt{4-x}}$ for $0 \le x \le 1$ and let R be the bounded region between the graph of f and the x-axis. Find the volume V of the solid obtained by revolving R about the x-axis.

First I'll draw the graph of f [note the domain is $(-\infty, 4)$] and shade the region R:



To rotate this around the x-axis, it will be practical to use washers



We thus set up the washer method:

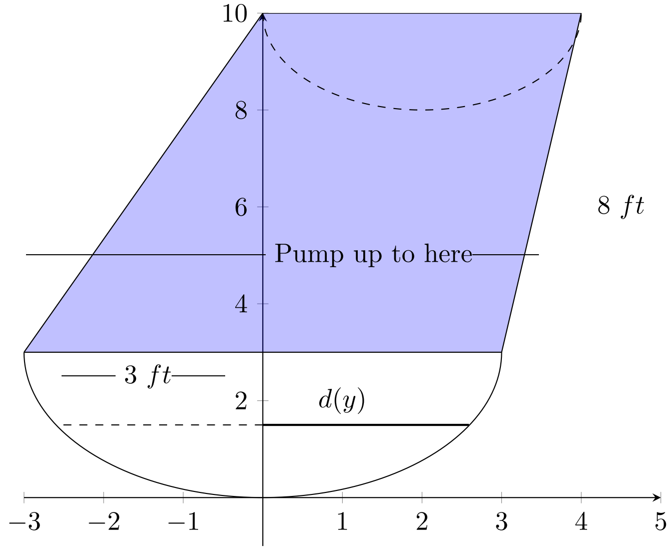
$$V = \pi \int_0^1 (f(x))^2 dx = \pi \int_0^1 (\frac{1}{\sqrt{4-x}})^2 dx = \pi \int_0^1 (\frac{1}{4-x}) dx$$

Letting $u = 4 - x$ so $du = -dx$, we have $u = 4 - 0 = 0$ for the new lower bound and $u = 4 - 1 = 3$ for the new upper. Then,

$$V = \pi \int_4^3 \frac{1}{u} - du = \pi \int_3^4 \frac{1}{u} du = \pi [\ln |u|]_3^4 = \pi (\ln 4 - \ln 3) = \pi \ln(\frac{4}{3}).$$

2 Review Problem 2 (from the Spring 2019 Final)

Suppose a pool has the shape of a half-cylinder 6 ft in diameter and 8 ft long. If the tank is full of water, write down the formula for the work necessary in order to pump the water up to a level 2 ft above the top of the tank. Draw a picture of the situation (DO NOT evaluate the integral).



We will use the formula $W = \int_a^b (62.5 \frac{lbf}{ft^3}) A(y)h(y)dy$, where $A(y)dy$ is the infinitesimal volume element and $h(y)$ is the distance over which the weight $62.5A(y)dy$ acts.

For us, $a = 0$, $b = 3$ are the water levels, and $h(y) = 5 - y$ since we need to lift the water up to the level $y = 5$ (e.g. the water at the top, at $y = 3$, needs only travel up 2 meters, i.e. $5 - 3$).

We still need the cross-sectional area $A(y)$ at each y. Note the cross sections are rectangles of length 8 and width $2d(y)$, thus we need to find an equation for $d(y)$.

Note that the equation of the full circle for the semicircle in the picture is $x^2 + (y-3)^2 = 3^2 = 9$, hence $x = d(y) = \pm\sqrt{9 - (y-3)^2}$. Since $x = d(y)$ is positive according to our diagram, $d(y) = \sqrt{9 - (y-3)^2}$ is the equation we need. Thus $A(y) = 2d(y) \cdot 8 = 16\sqrt{9 - (y-3)^2}$.

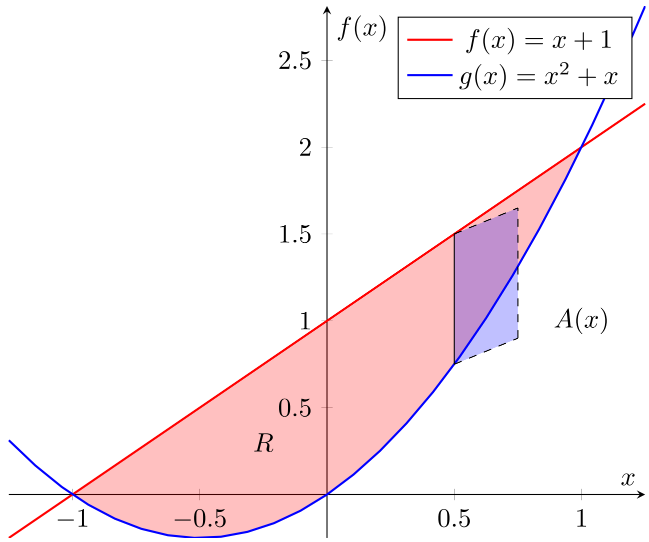
Finally,

$$W = \int_a^b (62.5 \tfrac{lbf}{ft^3}) A(y)h(y)dy = \int_{y=0}^{y=3} (62.5 \tfrac{lbf}{ft^3})(16\sqrt{9 - (y-3)^2})(5 - y)dy$$

3 Review Problem 3 (from the Spring 2018 Final)

Let R be the bounded region bounded by the graphs of $f(x) = x + 1$ and $g(x) = x^2 + x$. Let S be the solid region with R as its base, and with square cross-sections perpendicular to the x-axis. Calculate the volume V of S.

First I'll graph R:



We know our integral will be of the form $V = \int_{x=a}^{x=b} A(x)dx$ for some area function $A(x)$ for our cross-sections and intersection points $x = a$, $x = b$. We'll determine them.

Set up the equation $f(x) = g(x)$ to find the points of intersection corresponding to $x = a, x = b$ (after this, instead of graphing f and g one may do a trial point in between, say at $x = a$ to see whether $f(a) \leq g(a)$ or vice versa: this would indicate the top function on the interval (x_0, x_1) so you wouldn't have to graph it).

$$f(x) = g(x) \implies x + 1 = x^2 + x \implies x^2 = 1 \implies x = \pm 1.$$

Thus $a = -1$, $b = 1$.

Lastly, we need to determine $A(x)$ (the area of the blue square as a function of x). However, note that the side length of the square is $f(x) - g(x) = x + 1 - (x^2 + x) = 1 - x^2$, telling us

$$A(x) = (side)^2 = (f(x) - g(x))^2 = (1 - x^2)^2 = 1 - 2x^2 + x^4.$$

Finally,

$$\begin{aligned}
V = \int_{x=a}^{x=b} A(x)dx = \int_{x=-1}^{x=1} (1 - 2x^2 + x^4)dx &= [x - \frac{2x^3}{3} + \frac{x^5}{5}]_{-1}^{1} && (1)\\
&= 1 - \frac{2}{3} + \frac{1}{5} - (-1 - \frac{-2}{3} + \frac{-1}{5}) && (2)\\
&= 2(1 - \frac{2}{3} + \frac{1}{5}) = 2(\frac{1}{3} + \frac{1}{5}) && (3)\\
&= 2(\frac{5}{15} + \frac{3}{15}) && (4)\\
&= \frac{16}{15}. && (5)
\end{aligned}$$